МОДЕЛИ ДЛЯ ФОРМАЛЬНОЙ
АКСИОМАТИЧЕСКОЙ ТЕОРИИ ЗНАНИЯ Ξ

Аннотация

Определяется формальная аксиоматическая теория Ξ, представляющая собой философскую эпистемологию, и исследуется проблема ее логической непротиворечивости. Впервые выносятся на обсуждение такие качественно различные интерпретации аксиоматической системы Ξ, которые являются моделями для Ξ. С помощью этих моделей доказывается, что обсуждаемая формальная теория знания логически не противоречива.

Ключевые понятия:
формальная-аксиоматическая-теория; эпистемология; интерпретация; модель; непротиворечивость.

1. Introduction

A definition of the theory Ξ may be found in [15, 19–21]. During the oral presentation and discussion of Ξ at the World Congress on Universal Logic in Vichy, France, 2018, the logic consistency of Ξ was questioned. Moreover, some colleagues expressed the hypothesis that Ξ is inconsistent. Therefore, as in relation to philosophical epistemology, Ξ is a nontrivial novelty worthy of further development and systematical investigation, I have studied the consistency problem and submit results of the study below in this paper.

2. Definition of Ξ

For constructing a rigorous proof of logic consistency of the formal axiomatic epistemology theory Ξ it is indispensable to have a precise definition of that theory.
Therefore, the present paragraph 2 of this paper is aimed at making the reader acquainted with the rigorous formulation of $\Xi$ which can be found, for instance, in [19–21]. According to the definition given in these papers, the logically formalized axiomatic epistemology system $\Xi$ contains all symbols, expressions, formulae, axioms, and inference-rules of the classical propositional logic. Symbols $q$, $p$, $d$, … (called propositional letters) are elementary formulae of $\Xi$. Symbols $\alpha$, $\beta$, $\omega$, $\pi$, … (belonging to meta-language) stand for any formulae of $\Xi$. In general, the notion “formulae of $\Xi$” is defined as follows.

1) All propositional letters $q$, $p$, $d$, … are formulae of $\Xi$.
2) If $\alpha$ and $\beta$ are formulae of $\Xi$, then all such expressions of the object-language of $\Xi$, which possess logic forms $\neg \alpha$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$, $(\alpha \land \beta)$, $(\alpha \lor \beta)$, are formulae of $\Xi$ as well.
3) If $\alpha$ is a formula of $\Xi$, then $\Psi \alpha$ is a formula of $\Xi$ as well.
4) Successions of symbols (belonging to the alphabet of the object-language of $\Xi$) are formulae of $\Xi$, only if this is so owing to the above-given items (1) – (3) of the present definition.

The symbol $\Psi$ belonging to meta-language stands for any element of the set of modalities $\{\Box, K, A, E, S, T, F, P, Z, G, O, B, U, Y\}$. Symbol $\Box$ stands for the alethic modality “necessary”. Symbols $K, A, E, S, T, F, P, Z$, respectively, stand for modalities “agent knows that…”, “agent a-priori knows that…”, “agent a-posteriori knows that…”, “under some conditions in some space-and-time a person (immediately or by means of some tools) sensually perceives (has sensual verification) that…”, “it is true that…”, “agent believes that…”, “it is provable that…”, “there is an algorithm (a machine could be constructed) for deciding that…”.

Symbols $G, O, B, U, Y$, respectively, stand for modalities “it is (morally) good that…”, “it is obligatory that…”, “it is beautiful that…”, “it is useful that…”, “it is pleasant that…”. Meanings of the mentioned symbols are defined by the following schemes of own-axioms of epistemology system $\Xi$ which axioms are added to the axioms of classical propositional logic. Schemes of axioms and inference rules of the classical propositional logic are applicable to all formulae of $\Xi$ (including the additional ones).

Axiom scheme AX-1: $A\alpha \rightarrow (\Box \beta \rightarrow \beta)$.
Axiom scheme AX-2: $A\alpha \rightarrow (\Box (\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta))$.
Axiom scheme AX-3: $A\alpha \leftrightarrow (K\alpha \land (\Box \alpha \land \Box \neg S\alpha \land \Box (\beta \leftrightarrow \Omega \beta)))$.
Axiom scheme AX-4: $E\alpha \leftrightarrow (K\alpha \land (\neg \Box \alpha \lor \neg \Box \neg S\alpha \lor \neg \Box (\beta \leftrightarrow \Omega \beta)))$.

In AX-3 and AX-4, the symbol $\Omega$ (belonging to the meta-language) stands for any element of the set $R = \{\Box, K, T, F, P, Z, G, O, B, U, Y\}$. Let elements of $R$ be called “perfection-modalities” or simply “perfections”.

3. Models of/for $\Xi$

Above the axioms of $\Xi$ were defined by the axiom-schemes. Now first of all it is relevant to depart from the meta-language to the object-language, i.e. to move from the above axiom-schemes to the following axioms, respectively.

Axiom AX-1*: $Aq \rightarrow (\Box p \rightarrow p)$.
Axiom AX-2*: $Aq \rightarrow (\Box (q \rightarrow p) \rightarrow (\Box q \rightarrow \Box p))$.
Axiom AX-3*: $Aq \leftrightarrow (Kq \land (\Box q \land \Box \neg Sq \land \Box (p \leftrightarrow \Box p)))$. 
Axiom AX-4*: \( Eq \leftrightarrow (Kq \land (\neg \Box q \lor \neg \Box \neg Sq \lor \neg \Box (p \leftrightarrow \Box p))) \).

These axioms are obtained from the corresponding axiom-schemes by substituting: propositional letter \( q \) for \( a \); propositional letter \( p \) for \( \beta \); \( \Box \) for \( \Omega \). In this paper such interpretations of/for \( \Xi \) are considered in which all the axioms of \( \Xi \) are true. Now everything is prepared for defining and discussing interpretation-functions to be used for the demonstration of consistency.

Let \( \oplus \) stand for an element of the set of classical binary connectives \( \{\rightarrow, \leftrightarrow, \& , \lor\} \). Let \( @ \) stand for an element of the set of below-considered interpretation-functions \( \{¥, \nabla, €, £\} \). It is a *common* aspect of the below-given definitions of the interpretation-functions under consideration in this paper that, for any \( @, \oplus, \omega, \) and \( \pi \), it is true that:

1) \( @\neg \omega = \neg @\omega \);
2) \( @ (\omega \oplus \pi) = (@ \omega \oplus @ \pi) \).

Now let us move to *specific* aspects of the interpretation-function-definitions under review in this paper.

### 3.1. Interpretation ¥

3) ¥q = true.
4) ¥p = true.
5) ¥Aq = true.
6) ¥Kq = true.
7) ¥Eq = false.
8) ¥Sq = false.
9) For any \( \omega \), ¥\( \Box \omega \) = true: *everything is necessary*; this is an expression of such an extremely rationalistic a-priori-ism philosophy which can be extracted from writings of Spinoza [28] and Leibniz [11–14].

In the interpretation ¥, all the axioms of \( \Xi \) are true, consequently, \( \Xi \) has a model, hence \( \Xi \) is consistent.

### 3.2. Interpretation \( \nabla \)

3) \( \nabla q = true \).
4) \( \nabla p = true \).
5) \( \nabla Aq = false \).
6) \( \nabla Kq = true \).
7) \( \nabla Eq = true \).
8) \( \nabla Sq = true \).
9) For any \( \omega \), \( \nabla \Box \omega \) = false: *nothing is necessary*; this is an expression of such an extreme sensualism-and-empiricism philosophy which can be extracted from writings of Locke [22], Hume [7, 8], Berkeley [5], Mach [23, 24], Popper [26, 27], and Wittgenstein [29].

In the interpretation \( \nabla \) all the axioms of \( \Xi \) are true, consequently, \( \Xi \) has a model, hence \( \Xi \) is consistent.

### 3.3. Interpretation €

3) €q = true.
4) $\varepsilon p = \text{true.}$
5) $\varepsilon Aq = \varepsilon q.$
6) $\varepsilon Kq = \varepsilon q.$
7) $\varepsilon Eq = \varepsilon \neg q.$
8) $\varepsilon Sq = \varepsilon \neg q.$
9) For any $\omega$, $\varepsilon \square \omega = \varepsilon \omega.$

In the interpretation $\varepsilon$, all the axioms of $\Xi$ are true, consequently, $\Xi$ has a model, hence $\Xi$ is consistent.

3.4. Interpretation $\Xi$

3) $\xi q = \text{true.}$
4) $\xi p = \text{true.}$
5) $\xi Aq = \xi \neg q.$
6) $\xi Kq = \xi q.$
7) $\xi Eq = \xi q.$
8) $\xi Sq = \xi q.$
9) For any $\omega$, $\xi \square \omega = \xi \neg \omega.$

In the interpretation $\xi$, all the axioms of $\Xi$ are true, consequently, $\Xi$ has a model, hence $\Xi$ is consistent.

4. Formal proofs of philosophically interesting theorems in $\Xi$

Strictly speaking, here I mean not proofs of theorems but schemes of proofs of schemes of theorems. They are the following.

4.1. Theorem-scheme $(A\alpha \rightarrow (O\alpha \leftrightarrow G\alpha))$

Its formal proof (or, strictly speaking, scheme of proofs) in $\Xi$ is the following succession of formulae-schemes.
1) $A\alpha \leftrightarrow (K\alpha \& (\square\alpha \& \neg S\alpha \& \square(\beta \leftrightarrow \Omega\beta))$: axiom scheme AX-3.
2) $A\alpha$: assumption.
3) $K\alpha \& \square\alpha \& \neg S\alpha \& \square(\beta \leftrightarrow \Omega\beta)$: from 1 and 2 by propositional logic.
4) $\square(\beta \leftrightarrow \Omega\beta)$: from 3 by the rule of &-elimination.
5) $(\beta \leftrightarrow \Omega\beta)$: from 4 by the (limited) rule of $\square$-elimination.
6) $(\beta \leftrightarrow G\beta)$: from 5 by substituting $G$ for $\Omega$.
7) $(\beta \leftrightarrow O\beta)$: from 5 by substituting $O$ for $\Omega$.
8) $(O\beta \leftrightarrow \beta)$: from 7 by commutativity of $\leftrightarrow$.
9) $(O\beta \leftrightarrow G\beta)$: from 8 and 6 by transitivity of $\leftrightarrow$.
10) $A\alpha \leftarrow (O\beta \leftrightarrow G\beta)$: by 1–9.
11) $A\alpha \leftarrow (O\alpha \leftrightarrow G\alpha)$: from 10 by substituting $\alpha$ for $\beta$.
12) $\leftarrow (A\alpha \rightarrow (O\alpha \leftrightarrow G\alpha))$: from 11 by the rule of introduction of $\rightarrow$.

Here you are.

4.2. Theorem-scheme $(A\alpha \rightarrow (O\alpha \leftrightarrow \square\alpha))$

Its formal-proof-scheme is the following succession.
1) $A\alpha \leftrightarrow (Ka \& (\Box a \& \Box \neg S a \& \Box (\beta \leftrightarrow \Omega \beta))$: axiom scheme AX-3.
2) $A\alpha$: assumption.
3) $Ka \& (\Box a \& \Box \neg S a \& \Box (\beta \leftrightarrow \Omega \beta)$: from 1 and 2 by propositional logic.
4) $\Box (\beta \leftrightarrow \Omega \beta)$: from 3 by the rule of $\&$-elimination.
5) $(\beta \leftrightarrow \Omega \beta)$: from 4 by the (limited) rule of $\Box$-elimination.
6) $(\beta \leftrightarrow \Box \beta)$: from 5 by substituting $\Box$ for $\Omega$.
7) $(\beta \leftrightarrow O \beta)$: from 5 by substituting $O$ for $\Omega$.
8) $(O \beta \leftrightarrow \beta)$: from 7 by commutativity of $\leftrightarrow$.
9) $(O \beta \leftrightarrow \Box \beta)$: from 8 and 6 by transitivity of $\leftrightarrow$.
10) $A\alpha \mid \neg (O \beta \leftrightarrow \Box \beta)$: by 1–9.
11) $A\alpha \mid \neg (O \alpha \leftrightarrow \Box \alpha)$: from 10 by substituting $\alpha$ for $\beta$.
12) $\neg (A \alpha \rightarrow (O \alpha \leftrightarrow \Box \alpha))$: from 11 by the rule of introduction of $\rightarrow$.

Here you are.

Obviously, the above-given schemes of proofs are analogous; they are generalized by the following scheme of proofs of scheme of theorems in $\Xi$.

4.3. Theorem-scheme ($A\alpha \rightarrow (\Sigma \alpha \leftrightarrow \Omega \alpha)$)

For any $\Sigma$ and $\Omega$, it is provable in $\Xi$ that ($A\alpha \rightarrow (\Sigma \alpha \leftrightarrow \Omega \alpha)$), where the symbols $\Sigma$ and $\Omega$ (belonging to the meta-language) stand for any elements of the set $R = \{\Box, K, T, F, P, Z, G, O, B, U, Y\}$. (Elements of $R$ are called perfection-modalities.) The following succession of schemes of formulae is a scheme of proofs of for ($A\alpha \rightarrow (\Sigma \alpha \leftrightarrow \Omega \alpha)$) in $\Xi$.

1) $A\alpha \leftrightarrow (Ka \& (\Box a \& \Box \neg S a \& \Box (\beta \leftrightarrow \Omega \beta))$: axiom scheme AX-3.
2) $A\alpha \rightarrow (Ka \& (\Box a \& \Box \neg S a \& \Box (\beta \leftrightarrow \Omega \beta))$: from 1 by the rule of elimination of $\leftrightarrow$.
3) $A\alpha$: assumption.
4) $(Ka \& (\Box a \& \Box \neg S a \& \Box (\beta \leftrightarrow \Omega \beta))$: from 2 and 3 by modus ponens.
5) $\Box (\beta \leftrightarrow \Omega \beta)$: from 4 by the rule of elimination of $\&$.
6) $(\beta \leftrightarrow \Omega \beta)$: from 5 by the rule of elimination of $\Box$.
7) $(\alpha \leftrightarrow \Sigma \alpha)$: from 6 by substituting ($\alpha$ for $\beta$, and $\Sigma$ for $\Omega$).
8) $(\alpha \leftrightarrow \Omega \alpha)$: from 6 by substituting ($\alpha$ for $\beta$).
9) $(\Sigma \alpha \leftrightarrow \alpha)$: from 7 by commutativity of $\leftrightarrow$.
10) $(\Sigma \alpha \leftrightarrow \Omega \alpha)$: from 9 and 8 by transitivity of $\leftrightarrow$.
11) $A\alpha \mid \neg (\Sigma \alpha \leftrightarrow \Omega \alpha)$: by 1–10.
12) $\neg (A \alpha \rightarrow (\Sigma \alpha \leftrightarrow \Omega \alpha))$: from 11 by the rule of introduction of $\rightarrow$.

From the viewpoint of purely mathematical technique, the proof of ($A\alpha \rightarrow (\Sigma \alpha \leftrightarrow \Omega \alpha)$) is not interesting (too simple). But from the viewpoint of proper philosophy contents, the statement ($A\alpha \rightarrow (\Sigma \alpha \leftrightarrow \Omega \alpha)$) is very interesting and important. Various concrete philosophical interpretations (particular cases) of that statement are well-known as fundamental philosophical principles of the rationalism (a-priori-ism). For example, the following specific philosophical interpretations of the theorem-scheme ($A\alpha \rightarrow (\Sigma \alpha \leftrightarrow \Omega \alpha)$) are worth mentioning.

a) $A\alpha \rightarrow (G \alpha \leftrightarrow T \alpha)$: the rationalistic principle of optimism in ethics by N. Malebranche and G. W. Leibniz.
b) \( \alpha \rightarrow (T \alpha \leftrightarrow P \alpha) \): the rationalistic principle of *optimism in epistemology* by G. W. Leibniz and D. Hilbert. About modeling this principle see [15; 16; 19].

c) \( \alpha \rightarrow (P \alpha \leftrightarrow Z \alpha) \): the rationalistic principle of *mechanistic (algorithmic) optimism in epistemology* by R. Llull (Lullus), G. W. Leibniz, and A. A. Lovelace (Augusta Ada King-Noel, Countess of Lovelace).

d) \( \alpha \rightarrow (\alpha \leftrightarrow G \alpha) \): the rationalistic principle of equivalence between necessary being and (universal) goodness. This principle was expressed by some outstanding creators of Ancient-Roman-Law, for example, Ulpian, and some great theologians, for example, St. Tomas Aquinas [1; 2].

e) \( \alpha \rightarrow (G \alpha \leftrightarrow B \alpha) \): the principle of *kalokagathia* (Socrates, Xenophon, Plato, Aristotle [2; 3];

f) \( \alpha \rightarrow (G \alpha \leftrightarrow U \alpha) \): the principle of *utilitarianism* ethics (J. Bentham, J.-St. Mill [25]). About modeling this principle in \( \Xi \), see [17; 19].

g) \( \alpha \rightarrow (G \alpha \leftrightarrow Y \alpha) \): the principle of *hedonism* ethics (Aristippus, Epicurus). Modeling this principle in \( \Xi \) is discussed in [17; 19].

h) \( \alpha \rightarrow (B \alpha \leftrightarrow Y \alpha) \): the principle of *hedonism in aesthetics*;

i) \( \alpha \rightarrow (B \alpha \leftrightarrow U \alpha) \): the principle of *beauty of useful* (and *usefulness of beauty*).

j) \( \alpha \rightarrow (T \alpha \leftrightarrow U \alpha) \): the principle of *beauty as criterion of truth*. (W. Blake, P. A. M. Dirac).

k) \( \alpha \rightarrow (P \alpha \leftrightarrow B \alpha) \): the principle of *beauty as criterion of proof* (S. S. Averincev).

4.4. Theorem-scheme (\( \alpha \rightarrow (\Box \alpha \leftrightarrow \Box \Omega \alpha) \))

In addition to the above-said it is worth mentioning that the following succession of formula-schemes is a scheme of proofs (in \( \Xi \)) of the philosophically interesting theorem-scheme (\( \alpha \rightarrow (\Box \alpha \leftrightarrow \Box \Omega \alpha) \)), where \( \Omega \) takes values from the set \( \mathbb{R} \).

1) \( \alpha \rightarrow (K \alpha \& (\alpha \& \neg \neg S \alpha \& \Box (\beta \leftrightarrow \Omega \beta)) \): axiom scheme AX-3.

2) \( \alpha \): assumption.

3) \( K \alpha \& \neg \neg \alpha \& \neg \neg \neg S \alpha \& \Box (\beta \leftrightarrow \Omega \beta) \): from 1 and 2 by propositional logic.

4) \( \Box (\beta \leftrightarrow \Omega \beta) \): from 3 by the rule of &-elimination.

5) \( \Box (\alpha \leftrightarrow \Omega \alpha) \): from 4 by substituting \( \alpha \) for \( \beta \).

6) \( \alpha \rightarrow (\Box (\alpha \leftrightarrow \beta) \rightarrow (\alpha \leftrightarrow \beta)) \): theorem scheme.

7) \( \alpha \rightarrow (\Box (\alpha \leftrightarrow \Omega \alpha) \rightarrow (\alpha \leftrightarrow \Omega \alpha)) \): from 6 by substituting \( \Omega \alpha \) for \( \beta \).

8) \( \Box (\alpha \leftrightarrow \Omega \alpha) \rightarrow (\alpha \leftrightarrow \Omega \alpha) \): from 7 and 2 by modus ponens.

9) \( (\alpha \leftrightarrow \Omega \alpha) \rightarrow (\alpha \leftrightarrow \Omega \alpha) \): from 8 and 5 by modus ponens.

10) \( \neg (\alpha \rightarrow (\Box \alpha \leftrightarrow \Box \Omega \alpha)) \): by the rule of introduction of \( \neg \).

Here you are.

The theorem-scheme (\( \alpha \rightarrow (\Box \alpha \leftrightarrow \Box \Omega \alpha) \)) may be instantiated by the following nontrivial philosophical principles.

a) \( \alpha \rightarrow (\Box \alpha \leftrightarrow \Box G \alpha) \): the natural-law principle of *equivalence of necessary being and necessary positive-moral-value (necessary goodness)*, represented in works of Aristotle, Ulpian, and Aquinas. About this see [18; 19].
b) \( \text{Ap} \rightarrow (\square \text{p} \leftrightarrow \square \text{Op}) \): the natural-law principle of equivalence of necessary being and necessary norm (duty), represented in works of Cicero, I. Kant, and H. Kelsen. Of this principle see [18; 19].

From a) and b) it follows logically that \( \text{Ap} \rightarrow (\square \text{Op} \leftrightarrow \square \text{Ga}) \): the principle of equivalence of the normative (deontic) and the evaluative options of formulating the natural-law doctrine [18; 19].

Gödel’s necessitation rule does not belong to the set of inference rules of \( \Xi \). Nevertheless, it is easy to demonstrate in \( \Xi \) that under the condition that \( \text{Aa} \) (but not in general), the following (limited) inference-rule of necessitation is valid: “If \( \text{Aa} \vdash -\beta \), then \( \text{Aa} \vdash -\square \beta \)”. The following inference is a demonstration of this rule.

1. \( \text{Aa} \leftrightarrow (\text{Ka} \& (\square \alpha \& \square \neg \text{Sa} \& \square (\beta \leftrightarrow \Omega \beta))) \): axiom scheme AX-3.
2. \( \text{Aa} \): assumption.
3. \( \text{Ka} \& \square \alpha \& \neg \square \neg \text{Sa} \& \square (\beta \leftrightarrow \Omega \beta) \): from 1 and 2 by propositional logic.
4. \( \square (\beta \leftrightarrow \Omega \beta) \): from 3 by the rule of &-elimination.
5. \( (\beta \leftrightarrow \Omega \beta) \): from 4 by the (limited) rule of \( \square \) -elimination.
6. \( \text{Aa} \vdash -\beta \leftrightarrow \Omega \beta \): by 1–5.
7. \( \text{Aa} \vdash -\beta \leftrightarrow \square \beta \): from 6 by substituting \( \square \) for \( \Omega \).
8. \( \text{Aa} \vdash -\beta \): is given.
9. \( \text{Aa} \vdash -\square \beta \): from 7 and 8 by propositional logic.
10. If \( \text{Aa} \vdash -\beta \) then \( \text{Aa} \vdash -\square \beta \): by 1–9.

5. Conclusion

As there is at least one interpretation in which all axioms of \( \Xi \) are true (i.e. a model of/for \( \Xi \) exists), \( \Xi \) is consistent. Moreover, as all axioms of \( \Xi \) are true in both “absolutely opposite” interpretations, namely, the rationalism-a-priori-ism and the sensualism-empiricism ones, the two “opposites” are synthesized by \( \Xi \) without a logic contradiction.


MODELS FOR THE FORMAL AXIOMATIC EPISTEMOLOGY THEORY $\Xi$

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Annotation

The formal axiomatic theory in question is defined, and the problem of its logic consistency is investigated. For the first time such significantly different interpretations of the axiom system $\Xi$ are submitted which are models of/for $\Xi$. By means of these models it is demonstrated that the theory in question is consistent.

Key concepts:
formal-axiomatic-theory; epistemology; interpretation; model; consistency.